The Cauchy problem for dissipative wave equations with weighted nonlinear terms

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This is a joint work with Hidemitsu Wadade (Gifu University). We consider local and global energy solutions for the Cauchy problem of dissipative wave equations with nonlinear terms which have a singularity at the spatial origin. Let $n \geq 1$, $0 \leq s \leq 1$, $s < n/2$. Our Cauchy problem is given by

$$\begin{cases} (\partial_t^2 - \Delta + \partial_t)u(t,x) + \dfrac{f(u(t,x))}{|x|^s} = 0 \quad \text{for} \ \ (t,x) \in [0,\infty) \times \mathbb{R}^n \\ u(0,\cdot) = u_0(\cdot) \in H^1(\mathbb{R}^n), \quad \partial_t u(0,\cdot) = u_1(\cdot) \in L^2(\mathbb{R}^n), \end{cases} \tag{0.1}$$

where u_0, u_1 and f are real-valued functions, $\Delta := \sum_{j=1}^n \partial^2/\partial x_j^2$. Our main theorems are as follows, and they are proved by the use of the Caffarelli-Kohn-Nirenberg type inequality.

Theorem 0.1 *Let $n \geq 1$, $0 \leq s \leq 1$, $s < n/2$. Let $f(u) = \lambda |u|^{p-1}u$ or $f(u) = \lambda |u|^p$, and let p satisfy*

$$1 \leq p \begin{cases} < \infty & \text{if} \ \ n = 1, 2 \\ \leq 1 + \frac{2(1-s)}{n-2} & \text{if} \ \ n \geq 3. \end{cases} \tag{0.2}$$

Then we have the following results.

(1) For any u_0 and u_1, there exists $T = T(\|u_0\|_{H^1(\mathbb{R}^n)} + \|u_1\|_{L^2(\mathbb{R}^n)}) > 0$ and a unique solution u of (0.1) such that $u \in C_b([0,T), H^1(\mathbb{R}^n))$, $\partial_t u \in C_b([0,T), L^2(\mathbb{R}^n))$, and $\nabla_{t,x} u \in L^2((0,T) \times \mathbb{R}^n)$.

(2) If $\|u_0\|_{H^1(\mathbb{R}^n)} + \|u_1\|_{L^2(\mathbb{R}^n)}$ is sufficiently small, and $1 + 2(2-s)/n \leq p$, then the solution u of (1) is a global solution, namely, we are able to take $T = \infty$.

Theorem 0.2 *Let $n = 2$, $0 \leq s < 1$. Let $\alpha > 0$, $\lambda \in \mathbb{R}$. Let $f(u) = \lambda u(e^{\alpha u^2} - 1)$. Then we have the following results.*

(1) If $\|\nabla u_0\|_{L^2(\mathbb{R}^2)} < \{2\pi(1-s)/\alpha\}^{1/2}$, and $T > 0$ is sufficiently small, then (0.1) has a unique local solution u with $u \in C_b([0,T), H^1(\mathbb{R}^2))$ and $\partial_t u \in C_b([0,T), L^2(\mathbb{R}^2))$.

(2) If $\|u_0\|_{H^1(\mathbb{R}^2)} + \|u_1\|_{L^2(\mathbb{R}^2)}$ is sufficiently small, then the solution u of (1) is a global solution, namely, we are able to take $T = \infty$.