Hardy-type inequality for $0 < p < 1$ and hypodecreasing functions

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Let $0 < p < 1, \alpha \in \mathbb{R}$. We say that a function f is *hypodecreasing* with the parameters p, α if f is a function non-negative and measurable on $\mathbb{R}^n$ for which for some $M > 0$ for all $r > 0$ $\|f(x)|x|^{\alpha - \frac{n-1}{p}}\|_{L_p(B_r)} < \infty$ and

$$\|f\|_{L_1(B_r)} \le M r^{n - \frac{1}{p} - \alpha} \|f(x)|x|^{\alpha - \frac{n-1}{p}}\|_{L_p(B_r)}\,, \tag{1}$$

and we denote by $HD_p^\alpha(M)$ the space of all functions f non-negative and measurable on $\mathbb{R}^n$ for which inequality (1) holds for all $0 < r < \infty$. We also set $HD_p^\alpha = \bigcup_{M>0} HD_p^\alpha(M)$.

Each radially symmetric and non-increasing function belongs to the space HD_p^α for any $0 < p < 1$ and $\alpha \le n - \frac{1}{p}$. However, this is a much wider space, which also contains radially symmetric increasing functions if they are not growing too fast at the origin and at infinity. Moreover, the function $f(x) = \|\Delta_x^\sigma \varphi\|_{L_q(\mathbb{R}^n)}$, $x \in \mathbb{R}^n$, where $\sigma \in \mathbb{N}$, $\varphi \in L_q(\mathbb{R}^n)$, belongs to this space, which is quite important for applications.

We give sufficient conditions close to necessary ones in terms of spaces of hypodecreasing functions ensuring that the following stronger version of the Hardy-type inequality is satisfied for all functions f non-negative and measurable on $L_p(\mathbb{R}^n)$

$$\|t^\alpha (H(f\chi_{B_r}))(t)\|_{L_p(0,\infty)} \le N \|f(x)\chi_{B_r}(x)|x|^{\alpha - \frac{n-1}{p}}\|_{L_p(\mathbb{R}^n)} \tag{2}$$

for all $0 < r \le \infty$, where $N > 0$ is independent of f and r. (If $r = \infty$ this is the standard Hardy inequality.) Given $0 < p < 1$, $\alpha < n - \frac{1}{p}$ and $N > 0$, we denote by $H_p^\alpha(N)$ the space of all functions f non-negative and measurable on $\mathbb{R}^n$ for which inequality (2) is satisfied for all $0 < r \leq \infty$. We also set $H_p^\alpha = \bigcup_{N>0} H_p^\alpha(N)$.

Theorem. *Let* $0 < p < 1$ *and* $\alpha < n - \frac{1}{p}$. *Then for all* $\alpha < \beta \leq n - \frac{1}{p}$

$$HD_p^\beta \subset H_p^\alpha \subset HD_p^\alpha\,.$$

Detailed proofs, further results and applications are contained in [1], [2].

[1] Burenkov V.I., Senouci A., Tararykova T.V., Hardy-type inequality for $0 < p < 1$ and hypodecreasing functions. Eurasian Mathematical Journal **1** (2010), no. 3, 27-42.

[2] Burenkov V.I., Senouci A., Tararykova T.V., Equivalent quasi-norms involving differences and moduli of continuity. Complex Analysis and Elliptic Equations **55** (2010), no. 8-10, 759-769.