On a singular gap between Sobolev-Besov-Lorentz spaces in the limiting case

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The main purpose in this talk is to clarify the differences between the continuous embeddings on the Sobolev-Lorentz spaces and the Besov-Lorentz spaces in the limiting case. In the paper [1], the author proved the Gagliardo-Nirenberg type inequality on the critical Sobolev space $H_p^{\frac{n}{p}}(\mathbb{R}^n) := (I-\Delta)^{-\frac{n}{2p}} L_p(\mathbb{R}^n)$, $n \in \mathbb{N}$, $1 < p < \infty$, stating

$$\|u\|_{L_q} \leq C\, q^{\frac{1}{p'}} \|u\|_{L_p}^{\frac{p}{q}} \|(-\Delta)^{\frac{n}{2p}} u\|_{L_p}^{1-\frac{p}{q}} \tag{1}$$

for all $u \in H_p^{\frac{n}{p}}(\mathbb{R}^n)$ and all $p \leq q < \infty$. Our first aim is to extend (1) into the critical Sobolev-Lorentz space $H_{p_1,p_2}^{\frac{n}{p_1}}(\mathbb{R}^n) := (I-\Delta)^{-\frac{n}{2p_1}} L_{p_1,p_2}(\mathbb{R}^n)$, $1 < p_1 < \infty$, $1 \leq p_2 \leq \infty$, where $L_{p_1,p_2}(\mathbb{R}^n)$ denotes the Lorentz space. As a result, we obtain the following theorem:

Theorem 1. *(i) Let $n \in \mathbb{N}$ and $1 < p_1 < \infty$. Then there exists a positive constant C depending only on n and p_1 such that the inequality*

$$\|u\|_{L_{q_1,q_2}} \leq C\, q_1^{\frac{1}{p_2'}+\frac{1}{q_2}} \|u\|_{L_{p_1,p_2}}^{\frac{p_1}{q_1}} \|(-\Delta)^{\frac{n}{2p_1}} u\|_{L_{p_1,p_2}}^{1-\frac{p_1}{q_1}} \tag{2}$$

holds for all $u \in H_{p_1,p_2}^{\frac{n}{p_1}}(\mathbb{R}^n)$, where q_1, p_2 and q_2 are arbitrary exponents satisfying $p_1 \leq q_1 < \infty$ and $1 \leq p_2 \leq q_2 \leq \infty$.

(ii) The growth order $q_1^{\frac{1}{p_2'}+\frac{1}{q_2}}$ as $q_1 \to \infty$ in (2) is optimal. Indeed, (2) fails if we replace $q_1^{\frac{1}{p_2'}+\frac{1}{q_2}}$ by $q_1^{\frac{1}{p_2'}+\frac{1}{q_2}-\delta}$ for any $\delta > 0$.

Our next interest is to find the growth order for the embedding constants as well as to prove the Gagliardo-Nirenberg type inequalities for the critical Besov-Lorentz space. As a result, we obtain the following theorem:

Theorem 2. *Let $n \in \mathbb{N}$, $1 < p_1, r_1 < \infty$ and $1 \leq p_2, q_2, r_2, r_3 \leq \infty$. Then there exists a positive constant C such that the inequality*

$$\|u\|_{L_{q_1,q_2}} \leq C \max\left\{ q_1^{\frac{1}{r_3'}+\frac{1}{q_2}}, \left(\frac{1}{q_1-p_1}\right)^{\frac{1}{q_2}}, \left(\frac{1}{q_1-r_1}\right)^{\frac{1}{q_2}} \right\} \|u\|_{L_{p_1,p_2}}^{\frac{p_1}{q_1}} \|u\|_{\dot{B}_{r_1,r_2}^{\frac{n}{r_1},r_3}}^{1-\frac{p_1}{q_1}} \tag{3}$$

holds for all $u \in L_{p_1,p_2}(\mathbb{R}^n) \cap \dot{B}_{r_1,r_2}^{\frac{n}{r_1},r_3}(\mathbb{R}^n)$ and all q_1 with $\max\{p_1, r_1\} < q_1 < \infty$, where C is independent of q_1 and u.

Remark. *It is worth noting that the growth order $q_1^{\frac{1}{r_3'}+\frac{1}{q_2}}$ as $q_1 \to \infty$ in (3) depends only on q_2 and r_3, which implies that the second index r_2 for the Lorentz space does not play any role for the growth order for the embedding constant. Thus compared Theorem 1 with Theorem 2, we can observe a singular gap between the critical Sobolev-Lorentz space and the critical Besov-Lorentz space, that is, the former one is influenced by the second index for the Lorentz space in terms of the growth order as $q_1 \to \infty$, while the latter one is done by the third index for the Besov space.*

References

[1] T. Ozawa, *On critical cases of Sobolev's inequalities*, J. Funct. Anal. **127** (1995), 259–269.