A mathematical clue to the separation phenomena on the two-dimensional Navier-Stokes equation

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In this talk we give a mathematical clue to the separation phenomena (reverse flow phenomena). Let Ω be a domain with smooth boundary in R^2. The non-stationary Navier-Stokes equation is expressed as:

$$\partial_t u - \nu\Delta u + (u\cdot\nabla)u = -\nabla p,\ u|_{t=0} = u_0 \quad in \quad \Omega$$

with div $u = 0$ and $u|_{\partial\Omega} = 0$. We need to handle a shape of the boundary $\partial\Omega$ precisely, thus we set a parametrized smooth boundary $\varphi : (-\infty,\infty) \mapsto R^2$ as $|\partial_s\varphi(s)| = 1$, $\theta(\partial_s\varphi(s))$ is a decreasing function, $\cup_{-\infty<s<\infty}\varphi(s) \subset \partial\Omega$, where $\theta(w)$ is defined by a vector $w = r(\cos\tilde{\theta}, \sin\tilde{\theta})$, $\theta(w) := \tilde{\theta}$. For a technical sense, we need to assume that there are $\bar{s}_0$ and $\bar{s}_2$ (we set $\bar{s}_1 = 0$, $\bar{s}_0 < \bar{s}_1 < \bar{s}_2$) s.t. $\varphi(\bar{s}_0) = (0,0)$, $\theta(\partial_s\varphi(s))|_{s=\bar{s}_0} = 0$, $|\partial_s^2\varphi(s)|$ is monotone increasing for $s \in [\bar{s}_0, \bar{s}_1]$, $|\partial_s^2\varphi(s)| = 1/\delta$ for $s \in [\bar{s}_1, \bar{s}_2]$, where $1/\delta$ is a constant curvature of a part of obstacle boundary $\cup_{s\in[\bar{s}_1,\bar{s}_2]}\varphi(s)$.

Definition. (Normal coordinate.) For $s \in [\bar{s}_0, \bar{s}_2]$, let

$$\Phi(s,r) = \Phi_\varphi(s,r) := (\partial_s\varphi(s))^\perp r + \varphi(s).$$

We define $\perp$ as the upward direction.

Definition. (Normalized streamline for the initial data.) Let γ_a be in Ω near $\cup_{\bar{s}_0<s<\bar{s}_2}\varphi(s)$ which satisfies

$$\partial_s\gamma_a(s) = \left(\frac{u_0}{|u_0|}\right)(\gamma_a(s)), \quad \gamma_a(0) = a \in \Omega \quad near \quad \cup_{\bar{s}_0<s<\bar{s}_2}\varphi(s).$$

Definition. (Poincaré map.) For fixed s and s_1 sufficiently close to each other, let s_{min} be the minimum of $s' > 0$ for which there exists $\tau = \tau(s')$ such that $\Phi(s_1, \tau(s')) = \gamma_{\Phi(s,r)}(s')$. Let $L(r) = L_{s,s_1}(r) = \tau(s_{min})$.

Definition. (Parallel laminar flow.) The initial velocity u_0 near the boundary $\cup_{s'\in[\bar{s}_0,\bar{s}_2]}\varphi(s')$ is called: "Parallel laminar flow" iff $L(r)/r = 1$ near the boundary.

We set the initial datum $u_0 = (u_{0,1}(0,x_2), u_{0,2}(0,x_2))$ near the origin as follows:

$$u_{0,1}(0,x_2) = \alpha_1 x_2 - \frac{\alpha_2}{2}x_2^2 \quad and \quad u_{0,2}(0,x_2) = 0 \quad (\alpha_1,\alpha_2 > 0). \tag{1}$$

Theorem. Assume that $D_t u(t,x)|_{t=0} \to 0$ $(x \to x_0)$ for any $x_0 \in \partial\Omega$. If the initial datum satisfies (1) and has "Parallel laminar flow", then we have

$$\lim_{x\to\varphi(s)} \frac{\langle D_t u(x,t)|_{t=0}, \partial_s\varphi(s)\rangle}{\langle u_0(x), \partial_s\varphi(s)\rangle} = -\frac{\nu\alpha_2}{\delta\alpha_1} - \frac{\nu}{\delta^2} < 0 \quad for \quad s \in [\bar{s}_1, \bar{s}_2],$$

where $D_t u = \partial_t u + (u\cdot\nabla)u$.