Characterization of BMO via Banach function spaces

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The BMO norm is defined by

$$\|b\|_{\mathrm{BMO}} := \sup_{Q:\mathrm{cube}} \frac{1}{|Q|} \int_Q |b(x) - b_Q| \, dx,$$

where $|Q|$ is the Lebesgue measure of $Q \subset \mathbb{R}^n$ and b_Q is the mean value of the function b on Q. The well-known John–Nirenberg inequality implies that the BMO norm $\|b\|_{\mathrm{BMO}}$ is equivalent to

$$\sup_{Q:\mathrm{cube}} \frac{1}{\|\chi_Q\|_{L^p(\mathbb{R}^n)}} \|(b - b_Q)\chi_Q\|_{L^p(\mathbb{R}^n)}$$

for any constant $1 \le p < \infty$, where χ_Q is the characteristic function for Q. Replacing the constant p by a proper variable exponent $p(\cdot) : \mathbb{R}^n \to [1, \infty)$, we can also give the norm equivalent to $\|b\|_{\mathrm{BMO}}$ ([2, 4, 5]).

Now we consider the problem in the context of general function spaces. Let X be a Banach function space and define

$$\|b\|_{\mathrm{BMO}_X} := \sup_{Q:\mathrm{cube}} \frac{1}{\|\chi_Q\|_X} \|(b - b_Q)\chi_Q\|_X.$$

Ho [1] has initially proved that $\|b\|_{\mathrm{BMO}_X}$ is equivalent to $\|b\|_{\mathrm{BMO}}$, provided that the Hardy–Littlewood maximal operator is bounded on the associate space X'. Ho's proof is obtained as a by-product of some new results about atomic decomposition. In this talk, based on the recent paper [3] we give an another proof of the equivalence applying the Rubio de Francia algorithm.

References

[1] K.-P. Ho, *Atomic decomposition of Hardy spaces and characterization of BMO via Banach function spaces.* Anal. Math. **38** (2012), 173–185.

[2] M. Izuki, *Boundedness of commutators on Herz spaces with variable exponent.* Rend. Circ. Mat. Palermo (2) **59** (2010), 199–213.

[3] M. Izuki, *Another proof of characterization of BMO via Banach function spaces.* Rev. Un. Mat. Argentina, to appear.

[4] M. Izuki and Y. Sawano, *Variable Lebesgue norm estimates for BMO functions.* Czechoslovak Math. J. **62(137)** (2012), 717–727.

[5] M. Izuki, Y. Sawano and Y. Tsutsui, *Variable Lebesgue norm estimates for BMO functions. II.* Anal. Math. **40** (2014), 215–230.

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