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Scaling invariant Hardy inequalities of multiple logarithmic type on the whole space

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joint work with
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In this talk, we consider the Hardy type inequality of the logarithmic form:

$$\left\| \frac{f}{|x|(1+|\log|x||)} \right\|_{L_n(B_1)} \leq C\|f\|_{W_n^1(\mathbb{R}^n)}, \tag{$*$}$$

where $n \geq 2$ and $B_1 = \{x \in \mathbb{R}^n\,;\, |x| < 1\}$, see Edmunds-Triebel, Math.Nachr(1999) for instance. Our aim is to extend $(*)$ to the inequality having a homogeneous property. In fact, we show:

Theorem 1. *For $n \in \mathbb{N}$, $1 < \alpha < \infty$, $\max\{1, \alpha-1\} < \beta < \infty$ and $R > 0$, there holds*

$$\left(\int_{\mathbb{R}^n} \frac{\left| f(x) - f(R\frac{x}{|x|}) \right|^\beta}{\left| \log \frac{R}{|x|} \right|^\alpha} \frac{dx}{|x|^n} \right)^{\frac{1}{\beta}} \leq \frac{\beta}{\alpha-1} \left(\int_{\mathbb{R}^n} |x|^{\beta-n} \left| \log \frac{R}{|x|} \right|^{\beta-\alpha} \left| \frac{x}{|x|} \cdot \nabla f(x) \right|^\beta dx \right)^{\frac{1}{\beta}}. \tag{$**$}$$

Moreover, the constant $\frac{\beta}{\alpha-1}$ in the right-hand side of $(**)$ is best-possible.

It is easy to see that the special case of $\alpha = \beta = n \geq 2$ in $(**)$ yields $(*)$. Also since $(**)$ holds for all $R > 0$, we obtain

$$\sup_{R>0} \left(\int_{\mathbb{R}^n} \frac{\left| f(x) - f(R\frac{x}{|x|}) \right|^\beta}{\left| \log \frac{R}{|x|} \right|^\alpha} \frac{dx}{|x|^n} \right)^{\frac{1}{\beta}} \leq \frac{\beta}{\alpha-1} \sup_{R>0} \left(\int_{\mathbb{R}^n} |x|^{\beta-n} \left| \log \frac{R}{|x|} \right|^{\beta-\alpha} \left| \frac{x}{|x|} \cdot \nabla f(x) \right|^\beta dx \right)^{\frac{1}{\beta}}. \tag{$***$}$$

Note that both integrals in $(***)$ have a same scaling under $f_l(x) := f(\frac{x}{l})$ for $l > 0$. In this sense, $(***)$ is a logarithmic Hardy inequality of a homogeneous type differently from $(*)$. We can also establish a similar Hardy type inequality with multiple logarithmic weights preserving a homogeneous property.