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REMARKS ON NONLINEAR OPERATIONS ON MODULATION SPACES

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Let $F(f)$ be the composition of functions F and f. We consider the question "*If f belongs to some function space, does $F(f)$ belong to the same space again?*". The answers to this question for the Sobolev space $H^s_p(\mathbb{R}^n)$ and the Besov space $B^s_{p,q}(\mathbb{R}^n)$ are well known by Bony and Meyer when F is smooth enough. They developed the theory of paradifferential operators and proved that the answer is yes if $s > n/p$, F is smooth and $F(0) = 0$. The main part of this argument is to write the nonlinear operation $F(f)$ in the form of a linear operation

$$F(f) = M_{F,f}(x, D)f$$

where $M_{F,f}(x, D)$ is a pseudo-differential operator of the Hörmander class $S^0_{1,1}$. Since pseudo-differential operators of this class are H^s_p, $B^s_{p,q}$-bounded for $s > 0$, we get this affirmative answer.

In this talk, we will discuss this problem for the modulation space $M^s_{p,q}$ which is one of the function spaces introduced by Feichtinger in 1980's to measure the decaying and regularity property of a function or distribution in a way different from Sobolev spaces H^s_p or Besov spaces $B^s_{p,q}$. The main idea of the modulation space is to consider the space variable and the variable of its Fourier transform simultaneously, while they are treated independently in L^p-Sobolev spaces and Besov spaces. Fix a function $\varphi \in C^\infty_0(\mathbb{R}^n) \setminus \{0\}$ (called the *window function*). Then the short-time Fourier transform $V_\varphi f$ of f with respect to φ is defined by

$$V_\varphi f(x, \xi) = \int_{\mathbb{R}^n} f(t)\, \overline{\varphi(t - x)}\, e^{-i\xi\cdot t}\, dt \quad (x, \xi \in \mathbb{R}^n).$$

Then modulation spaces $M^s_{p,q}$ are defined by the norm

$$\|f\|_{M^s_{p,q}} = \Big\| \|\langle\xi\rangle^s V_\varphi f(x, \xi)\|_{L^p_x} \Big\|_{L^q_\xi}, \quad \langle\xi\rangle = (1 + |\xi|^2)^{1/2}.$$

Unfortunately the answer to the question

$$F(t) \in C^\infty,\ f \in M^s_{p,q} \implies F(f) \in M^s_{p,q}\ ?$$

is unclear except for some restricted cases (when F has the analyticity). The main obstacle is the fact that pseudo-differential operators of class $S^0_{1,\delta}$ $(0 \le \delta < 1)$ have exotic mapping property on modulation spaces and it is hard to expect that pseudo-differential operators of class $S^0_{1,1}$ is bounded on $M^s_{p,q}(\mathbb{R}^n)$. On the other hand, it is know that the answer is yes if $s > n(1 - 1/q)$ and $1 < q \le \infty$, ($s \ge n(1 - 1/q)$ when $q = 1$), F is entire and $F(0) = 0$. Recently Bhimani-Ratnakumar and Kobayashi-Sato gave an negative answer for $M^0_{p,1}$, and proved that the answer is yes if and only if F is real analytic. We will discuss the general case $M^s_{p,q}$ with $q > 1$ when F is smooth but not necessarily analytic.