REMARKS ON NONLINEAR OPERATIONS
ON MODULATION SPACES

MITSURU SUGIMOTO (NAGOYA UNIVERSITY)

Let $F(f)$ be the composition of functions $F$ and $f$. We consider the question “If $f$ belongs to some function space, does $F(f)$ belong to the same space again?”. The answers to this question for the Sobolev space $H^s_p(\mathbb{R}^n)$ and the Besov space $B^{s}_{p,q}(\mathbb{R}^n)$ are well known by Bony and Meyer when $F$ is smooth enough. They developed the theory of paradifferential operators and proved that the answer is yes if $s > n/p$, $F$ is smooth and $F(0) = 0$. The main part of this argument is to write the nonlinear operation $F(f)$ in the form of a linear operation

$$ F(f) = M_{F,f}(x, D)f $$

where $M_{F,f}(x, D)$ is a pseudo-differential operator of the Hörmander class $S^{0}_{1,1}$. Since pseudo-differential operators of this class are $H^s_p$, $B^{s}_{p,q}$-bounded for $s > 0$, we get this affirmative answer.

In this talk, we will discuss this problem for the modulation space $M^s_{p,q}$ which is one of the function spaces introduced by Feichtinger in 1980’s to measure the decaying and regularity property of a function or distribution in a way different from Sobolev spaces $H_{p}^s$ or Besov spaces $B^{s}_{p,q}$. The main idea of the modulation space is to consider the space variable and the variable of its Fourier transform simultaneously, while they are treated independently in $L^p$-Sobolev spaces and Besov spaces. Fix a function $\varphi \in C_0^\infty(\mathbb{R}^n) \setminus \{0\}$ (called the window function). Then the short-time Fourier transform $V_{\varphi,f}$ of $f$ with respect to $\varphi$ is defined by

$$ V_{\varphi,f}(x, \xi) = \int_{\mathbb{R}^n} f(t) \overline{\varphi(t-x)} e^{-i\xi \cdot t} \, dt \quad (x, \xi \in \mathbb{R}^n). $$

Then modulation spaces $M^s_{p,q}$ are defined by the norm

$$ \|f\|_{M^s_{p,q}} = \|\langle \xi \rangle^s V_{\varphi,f}(x, \xi)\|_{L^p_x(L^q_\xi)} \quad (\langle \xi \rangle = (1 + |\xi|^2)^{1/2}. $$

Unfortunately the answer to the question

$$ F(t) \in C^\infty, \ f \in M^s_{p,q} \implies F(f) \in M^s_{p,q} $$

is unclear except for some restricted cases (when $F$ has the analyticity). The main obstacle is the fact that pseudo-differential operators of class $S_{1,1}^0$ ($0 \leq \delta < 1$) have exotic mapping property on modulation spaces and it is hard to expect that pseudo-differential operators of class $S_{1,1}^0$ is bounded on $M^s_{p,q}(\mathbb{R}^n)$. On the other hand, it is know that the answer is yes if $s > n(1-1/q)$ and $1 < q \leq \infty$, ($s \geq n(1-1/q)$ when $q = 1$), $F$ is entire and $F(0) = 0$. Recently Bhimani-Ratnakumar and Kobayashi-Sato gave an negative answer for $M^0_{p,1}$, and proved that the answer is yes if and only if $F$ is real analytic. We will discuss the general case $M^s_{p,q}$ with $q > 1$ when $F$ is smooth but not necessarily analytic.