Harmonic Analysis and its Applications in Tokyo 2017
http://enakai.sci.ibaraki.ac.jp/haat2017/

Finite energy for the Navier-Stokes equations and Liouville-type theorems

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This talk is based on a joint work with Professor Hideo Kozono (Waseda University) and Professor Yuta Wakasugi (Ehime University).

We consider the Cauchy problem for the Navier-Stokes equations

$$\begin{cases} v_t - \Delta v + (v\cdot\nabla)v + \nabla p = 0, & (x,t) \in \mathbb{R}^n \times (0,T), \\ \operatorname{div} v = 0, & (x,t) \in \mathbb{R}^n \times (0,T), \\ v(x,0) = v_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{0.1}$$

Here $v = v(x,t) = (v_1(x,t), \ldots, v_n(x,t))$ and $p = p(x,t)$ denote the velocity and the pressure, respectively. Also, $v_0(x) = (v_0^1(x), \ldots, v_0^n(x))$ stands for the given initial velocity.

Serrin, Shinbrot and Taniuchi proved the conditions for the energy identity of weak solutions of (0.1). Here we give another condition which ensures the energy identity of a smooth solution and as its application, we also give a Liouville-type result. Our first main result is the following.

Theorem 1. *Let $n \geq 2$, $v_0 \in L^2_\sigma(\mathbb{R}^n)$ and let (v,p) be a smooth solution of* (0.1). *Assume that there exist q_1, q_2, r_1, r_2 satisfying*

$$3 \leq q_1 \leq \frac{3n}{n-1},\ 3 \leq r_1 \leq \infty \quad \textit{and} \quad (q_1, r_1) \neq \left(\frac{3n}{n-1}, \infty\right), \tag{0.2}$$

$$2 \leq q_2 \leq \frac{2n}{n-2},\ 2 \leq r_2 \leq \infty \quad \textit{and} \quad \begin{cases} (q_2, r_2) \neq \left(\frac{2n}{n-2}, \infty\right) & (n \geq 3), \\ q_2 \neq \infty & (n = 2) \end{cases} \tag{0.3}$$

such that $v \in L^3(0,T; L^{q_1,r_1}(\mathbb{R}^n)) \cap L^2(0,T; L^{q_2,r_2}(\mathbb{R}^n))$. Assume also that the pressure p satisfies $p \in L^\infty(\mathbb{R}^n \times (0,T))$. Then, we have that

$$v \in L^\infty(0,T; L^2_\sigma(\mathbb{R}^n)) \cap L^2(0,T; \dot{H}^1_\sigma(\mathbb{R}^n))$$

and that

$$\|v(t)\|_{L^2}^2 + 2\int_0^t \|\nabla v(\tau)\|_{L^2}^2 d\tau = \|v_0\|_{L^2}^2$$

for all $t \in (0,T)$.

An immediate consequence of this theorem is the following Liouville-type theorem.

Corollary 2. *Let $n \geq 2$, and let $v_0 \equiv 0$ in $\mathbb{R}^n$. Suppose that v is a smooth solution of (0.1) with its associted pressure p. If $v \in L^3(0,T; L^{q_1,r_1}(\mathbb{R}^n)) \cap L^2(0,T; L^{q_2,r_2}(\mathbb{R}^n))$ for such (q_1,r_1) and (q_2,r_2) as in (0.2) and (0.3), respectively, and if $p \in L^\infty(\mathbb{R}^n \times (0,T))$, then it holds that $v(x,t) \equiv 0$ on $\mathbb{R}^n \times (0,T)$.*

We also treat the marginal case of (0.2) and (0.3) and get the condition for finite energy of solutions and get the corresponding Liouville-type theorem.

If time allows, we also handle the equation for vorticity in 2D unbounded domains. Under some asymptotic behavior at infinity and some conditions at the boundary, we prove that the vorticity are square integrable.