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On a maximizing problem associated with the Sobolev type embedding in the whole space with lower order terms

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In this talk, we consider the attainability of a maximizing problem

$$D_\alpha = D_\alpha(N,p,q,a,b) = \sup_{\|\nabla u\|^a_{L^p(\mathbb{R}^N)}+\|u\|^b_{L^p(\mathbb{R}^N)}=1} \left(\|u\|^p_{L^p(\mathbb{R}^N)} + \alpha\|u\|^q_{L^q(\mathbb{R}^N)}\right),$$

where $N \geq 2$, $1 < p < N$, $p < q < \frac{Np}{N-p}$, $a, b, \alpha > 0$. The existence and non-existence of a maximizer for D_α is closely related to the exponents α, a and b. The boundedness of the value D_α is guaranteed by the following Sobolev inequality :

$$\|u\|^p_{L^q(\mathbb{R}^N)} \leq S\left(\|\nabla u\|^p_{L^p(\mathbb{R}^N)} + \|u\|^p_{L^p(\mathbb{R}^N)}\right),$$

where $N \geq 2$, $1 < p < N$, $p \leq q \leq \frac{Np}{N-p}$ and S is the best-constant given by

$$S := \sup_{\|\nabla u\|^p_{L^p(\mathbb{R}^N)}+\|u\|^p_{L^p(\mathbb{R}^N)}=1} \|u\|^p_{L^q(\mathbb{R}^N)}.$$

The variational problem associated with S suffers from a non-compactness. Indeed, (i) for $p < q < \frac{Np}{N-p}$, the embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is non-compact due to the action of the spacial translation $u(\cdot) \mapsto u(\cdot - y)$ for $y \in \mathbb{R}^N$; (ii) for $q = \frac{Np}{N-p}$, the non-compactness of the embedding comes from the invariance under the scaling $u_\lambda(\cdot) := \lambda^{\frac{N-p}{p}} u(\lambda\cdot)$ for $\lambda > 0$ keeping the $L^{\frac{Np}{N-p}}$-norm invariant ; (iii) for $q = p$, the non-compactness of the embedding comes from the invariance under the scaling $u_\lambda(\cdot) := \lambda^{\frac{N}{p}} u(\lambda\cdot)$ for $\lambda > 0$ keeping the L^p-norm invariant.

The non-compactness of (iii) stems from that of $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^N)$. One of the purposes of this talk is to investigate the effect of "non-compact" term $\|\cdot\|_{L^p(\mathbb{R}^N)}$ on the existence and non-existence of a maximizer for D_α. Another purpose in this talk is to clarify the existence and non-existence of a maximizer for D_α in terms of the exponents a, b, α. In particular, the valance of the values of a and b influences the structure of the current variational problem. Our main result is stated as follows :

Theorem 1. *Define the value α_* and the best-constant of the Gagliardo-Nirenberg inequality B by*

$$\alpha_* := \inf_{\|\nabla u\|^a_{L^p(\mathbb{R}^N)}+\|u\|^b_{L^p(\mathbb{R}^N)}=1} \frac{1 - \|u\|^p_{L^p(\mathbb{R}^N)}}{\|u\|^q_{L^q(\mathbb{R}^N)}} \quad \text{and} \quad B := \sup_{u \in W^{1,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|^q_{L^q(\mathbb{R}^N)}}{\|\nabla u\|^{\frac{N(q-p)}{p}}_{L^p(\mathbb{R}^N)} \|u\|^{q-\frac{N(q-p)}{p}}_{L^p(\mathbb{R}^N)}}.$$

Then the followings hold true :
(i) If $a > \frac{N(q-p)}{p}$, then $\alpha_* = 0$, and D_α is attained for all $\alpha > 0$.
(ii) Assume $a = \frac{N(q-p)}{p}$. (ii)-(a1) Let $\frac{p(2N-p)}{2(N-p)} < q < \frac{Np}{N-p}$. (ii)-(a1)-(b1) Let $b \geq b_0 := \frac{(q-p)(N-p)}{p} - \frac{Np-(N-p)q}{p} (> 0)$. Then $\alpha_* = \frac{p}{bB}$, and D_α is attained for all $\alpha > \alpha_*$, while D_α is not attained for all $\alpha \leq \alpha_*$. (ii)-(a1)-(b2) Let $b < b_0$. Then $0 < \alpha_* < \frac{p}{bB}$, and D_α is attained for all $\alpha \geq \alpha_*$, while D_α is not attained for all $\alpha < \alpha_*$. (ii)-(a2) Let $p < q \leq \frac{p(2N-p)}{2(N-p)}$. Then $\alpha_* = \frac{p}{bB}$, D_α is attained for all $\alpha > \alpha_*$, while D_α is not attained for all $\alpha \leq \alpha_*$.
(iii) Assume $a < \frac{N(q-p)}{p}$. Then $\alpha_* > 0$, D_α is attained for all $\alpha \geq \alpha_*$, while D_α is not attained for all $\alpha < \alpha_*$.

The content in this talk is a joint work with Prof. Michinori Ishiwata in Osaka University.