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A WEAK TYPE OF NORM INFLATION FOR SOLUTIONS OF THE INCOMPRESSIBLE 2D EULER EQUATIONS NEAR THE CRITICAL BESOV SPACE $B^2_{2,1}$

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We study the Cauchy problem for the Euler equations of an incompressible and inviscid fluid

$$u_t + u\cdot\nabla u = -\nabla p, \quad \operatorname{div} u = 0, \qquad t\geq 0,\ x\in\mathbb{R}^n, \tag{0.1}$$
$$u(0)=u_0,$$

where $u=u(t,x)$ and $p=p(t,x)$ denote the velocity field and the pressure function of the fluid respectively. We prove an ill-posedness result for the vorticity equations that involves a family of Besov spaces. Although this result is weaker than instantaneous blowup described by Bourgain and Li (2015) our methods can be applied in the borderline end-point spaces such as $B^2_{2,1}(\mathbb{R}^2)$ which lie just outside the range of the spaces considered in BL. Recall that existence and uniqueness results for (0.1) in $B^2_{2,1}(\mathbb{R}^2)$ are already known, cf. e.g. Vishik (1998) or Chae (2004).

Theorem 0.1. *Let $M_j\nearrow\infty$ be an increasing sequence of positive numbers. There exists a sequence of smooth rapidly decaying initial data $\{\tilde u_{0,j}\}_{j=1}^\infty$ and two sequences of indices $\{r_j\}_{j=1}^\infty$ and $\{q_j\}_{j=1}^\infty$ with $r_j\to 2$ and $q_j\to 1$ such that*

$$\|\tilde u_{0,j}\|_{B^2_{r_j,q_j}}\lesssim 1 \quad \textit{and} \quad \|\tilde u_j(t)\|_{B^2_{r_j,q_j}} > M_j \quad \textit{for some } 0<t<M_j^{-3}.$$

Remark 0.2. It is of interest to compare this result with the a'priori estimates for local solutions in Besov norms. If we let $C_j>0$ denote the best constant in the Besov version of the Sobolev embedding

$$\|\nabla\tilde u_j\|_\infty \leq C_j\|\tilde u_j\|_{B^2_{r_j,q_j}} \tag{0.2}$$

(cf. Remark 2.1 in Chae (2004)) then Theorem 0.1 implies that $C_j\nearrow\infty$ or else we get a contradiction with the standard bound $\sup_{0\leq t\leq T_j}\|\tilde u_j(t)\|_{B^2_{r_j,q_j}}\leq C_{0,j}\|\tilde u_{0,j}\|_{B^2_{r_j,q_j}}$ where $C_{0,j}$ depends on C_j. On the other hand, for $1\leq p<\infty$ and $1\leq q\leq\infty$, we also have the estimate

$$\|\nabla\tilde u_\infty\|_\infty\leq C_\infty\|\tilde u_\infty\|_{B^{1+2/p}_{p,q}} \tag{0.3}$$

for some finite constant $C_\infty>0$ if and only if $q=1$ (see Theorem 11.4, p.170 in "The structure of functions" written by Triebel). This suggests that the dependence of the constants in (0.2) and (0.3) on the Besov parameters is not continuous. To the best of our knowledge this dependence has not been investigated in the literature.

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