A WEAK TYPE OF NORM INFLATION FOR SOLUTIONS OF
THE INCOMPRESSIBLE 2D EULER EQUATIONS NEAR THE
CRITICAL BESOV SPACE $B_{2,1}^2$

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We study the Cauchy problem for the Euler equations of an incompressible and
inviscid fluid
\begin{equation}
\tag{0.1}
u_t + u \cdot \nabla u = -\nabla p, \quad \text{div } u = 0, \quad t \geq 0, \quad x \in \mathbb{R}^n, \\
u(0) = u_0,
\end{equation}
where $u = u(t, x)$ and $p = p(t, x)$ denote the velocity field and the pressure function of
the fluid respectively. We prove an ill-posedness result for the vorticity equations
that involves a family of Besov spaces. Although this result is weaker than instantan-
eous blowup described by Bourgain and Li (2015) our methods can be applied in
the borderline end-point spaces such as $B_{2,1}^2(\mathbb{R}^2)$ which lie just outside the range of
the spaces considered in BL. Recall that existence and uniqueness results for (0.1)
in $B_{2,1}^2(\mathbb{R}^2)$ are already known, cf. e.g. Vishik (1998) or Chae (2004).

**Theorem 0.1.** Let $M_j \not\to \infty$ be an increasing sequence of positive numbers. There
exists a sequence of smooth rapidly decaying initial data $\{u_{0,j}\}_{j=1}^\infty$ and two sequences
of indices $\{r_j\}_{j=1}^\infty$ and $\{q_j\}_{j=1}^\infty$ with $r_j \to 2$ and $q_j \to 1$ such that
$$\|u_{0,j}\|_{B_{r_j,q_j}^2} \lesssim 1 \quad \text{and} \quad \|u_j(t)\|_{B_{r_j,q_j}^2} > M_j \quad \text{for some } 0 < t < M_j^{-3}.$$ 

**Remark 0.2.** It is of interest to compare this result with the a’priori estimates for
local solutions in Besov norms. If we let $C_j > 0$ denote the best constant in the
Besov version of the Sobolev embedding
\begin{equation}
\tag{0.2}
\|\nabla a_j\|_{\infty} \leq C_j \|a_j\|_{B_{r_j,q_j}^2}
\end{equation}
(cf. Remark 2.1 in Chae (2004)) then Theorem 0.1 implies that $C_j \not\to \infty$ or
else we get a contradiction with the standard bound $\sup_{0 \leq t \leq T_j} \|a_j(t)\|_{B_{r_j,q_j}^2} \leq C_{0,j} \|a_{0,j}\|_{B_{r_j,q_j}^2}$ where $C_{0,j}$ depends on $C_j$. On the other hand, for $1 \leq p < \infty$ and
$1 \leq q \leq \infty$, we also have the estimate
\begin{equation}
\tag{0.3}
\|\nabla u_{\infty}\|_{\infty} \leq C_\infty \|u_{\infty}\|_{B_{p,1}^{1+2/p}}
\end{equation}
for some finite constant $C_\infty > 0$ if and only if $q = 1$ (see Theorem 11.4, p.170 in
“The structure of functions” written by Triebel). This suggests that the dependence
of the constants in (0.2) and (0.3) on the Besov parameters is not continuous. To the
best of our knowledge this dependence has not been investigated in the literature.

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